

# A note on two linear forms

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## 1. Diophantine exponents.

Let  $\theta_1, \theta_2$  be real numbers such that

$$1, \theta_1, \theta_2 \text{ are linearly independent over } \mathbb{Z}. \quad (1)$$

We consider linear form

$$L(\mathbf{x}) = x_0 + x_1\theta_1 + x_2\theta_2, \quad \mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3.$$

By  $|\mathbf{z}|$  we denote the Euclidean length of a vector  $\mathbf{z} = (z_0, z_1, z_2) \in \mathbb{R}^3$ . Let

$$\hat{\omega} = \hat{\omega}(\theta_1, \theta_2) = \sup \left\{ \gamma : \limsup_{t \rightarrow \infty} \left( t^\gamma \min_{0 < |\mathbf{x}| \leq t} |L(\mathbf{x})| \right) < \infty \right\} \quad (2)$$

be the uniform Diophantine exponent for the linear form  $L$ .

We consider another linear form  $P(\mathbf{x})$ . The main result of the present paper is as follows.

**Theorem 1.** *Suppose that linear forms  $L(x)$  and  $P(x)$  are independent and the exponent  $\hat{\omega}$  for the form  $L$  are defined in (2). Then for the Diophantine exponent*

$$\omega_{LP} = \sup \left\{ \gamma : \text{there exist infinitely many } \mathbf{x} \in \mathbb{Z}^3 \text{ such that } |L(\mathbf{x})| \leq |P(\mathbf{x})| \cdot |\mathbf{x}|^{-\gamma} \right\}$$

we have a lower bound

$$\omega_{LP} \geq \hat{\omega}^2 - \hat{\omega} + 1.$$

**Remark.** Of course in the definition (2) and in Theorem 1 instead of the Euclidean norm  $|\mathbf{x}|$  we may consider the value  $\max_{i=1,2} |x_i|$  as it was done by the most of authors.

Consider a real  $\theta$  which is not a rational number and not a quadratic irrationality. Define

$$\omega_* = \omega_*(\theta) = \sup \{ \gamma : \text{there exist infinitely many algebraic numbers } \xi \text{ of degree } \leq 2$$

$$\text{such that } |\theta - \xi| \leq H(\xi)^{-\gamma} \}$$

(here  $H(\xi)$  is the maximal value of the absolute values of the coefficients for canonical polynomial to  $\xi$ ). Then for linear forms

$$L(\mathbf{x}) = x_0 + x_1\theta + x_2\theta^2, \quad P(\mathbf{x}) = x_1 + 2x_2\theta$$

one has

$$\omega_* \geq \omega_{LP}. \quad (3)$$

So Theorem 1 immediately leads to the following corollary.

**Theorem 2.** *For a real  $\theta$  which is not a rational number and not a quadratic irrationality one has*

$$\omega_* \geq \hat{\omega}^2 - \hat{\omega} + 1 \quad (4)$$

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with  $\hat{\omega} = \hat{\omega}(\theta, \theta^2)$ .

## 2. Some history.

In 1967 H. Davenport and W. Schmidt [2] (see also Ch. 8 from Schmidt's book [11]) proved that for any two independent linear forms  $L, P$  there exist infinitely many integer points  $\mathbf{x}$  such that

$$|L(\mathbf{x})| \leq C|P(\mathbf{x})| |\mathbf{x}|^{-3},$$

with a positive constant  $C$  depending on the coefficients of forms  $L, P$ . From this result they deduced that for any real  $\theta$  which is not a rational number and not a quadratic irrationality the inequality

$$|\theta - \xi| \leq C_1 H(\xi)^{-3}$$

has infinitely many solutions in algebraic  $\xi$  of degree  $\leq 2$ .

We see that for any two pairs of forms one has  $\omega_{LP} \geq 3$ . But from the Minkowski convex body theorem it follows that under the condition (1) one has  $\hat{\omega} \geq 2$ . Moreover

$$\min_{\hat{\omega} \geq 2} (\hat{\omega}^2 - \hat{\omega} + 1) = 3.$$

So our Theorems 1,2 may be considered as generalizations of Davenport-Schmidt's results.

Later Davenport and Schmidt generalized their theorems to the case of several linear forms [3]. In the next paper [4] they showed that the value of the uniform exponent for *simultaneous* approximations to any point  $(\theta, \theta^2)$  is not greater than  $\frac{\sqrt{5}-1}{2}$ . This together with Jarník's transference equality (see [5]) leads to the bound  $\hat{\omega} \leq \frac{3+\sqrt{5}}{2}$  which holds for all linear forms with coefficients of the form  $\theta, \theta^2$ . So for a linear form with coefficients  $\theta, \theta^2$  one has

$$2 \leq \hat{\omega} \leq \frac{3 + \sqrt{5}}{2}. \quad (5)$$

D. Roy [9, 10] showed that the set of values  $\hat{\omega}$  for linear forms under our consideration form a dense set in the segment (5). Moreover he constructed a countable set of numbers  $\theta$  such that

$$\hat{\omega}(\theta, \theta^2) = \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad \omega_*(\theta) = 3 + \sqrt{5}.$$

This shows that our bound (4) from Theorem 2 is optimal in the right endpoint of the segment (5), namely for  $\hat{\omega} = \frac{3+\sqrt{5}}{2}$ .

Other results on approximation by algebraic numbers are discussed in W. Schmidt's book [11], in wonderful book by Y. Bugeaud [1] and in M. Waldschmidt's survey [12].

Our proof of Theorem 1 generalizes ideas from [2, 3, 4] and uses Jarník's inequalities [6, 7].

## 3. Minimal points.

In the sequel we may suppose that  $\hat{\omega} > 2$  as the case  $\hat{\omega} = 2$  follows from Davenport-Schmidt's theorem (in this case our Theorem 1 claims that  $\omega_{LP} \geq 3$ ). We take  $\alpha < \hat{\omega}$  close to  $\hat{\omega}$  so that  $\alpha > 2$ .

A vector  $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$  is defined to be a *minimal point* (or *best approximation*) if

$$\min_{\mathbf{x}': 0 < |\mathbf{x}'| \leq |\mathbf{x}|} |L(\mathbf{x}')| = L(\mathbf{x}).$$

As  $1, \theta_1, \theta_2$  are linearly independent, all the minimal points form a sequence  $\mathbf{x}_\nu = (x_{0,\nu}, x_{1,\nu}, x_{2,\nu})$ ,  $\nu = 1, 2, 3, \dots$  such that for  $X_\nu = |\mathbf{x}_\nu|$ ,  $L_\nu = L(\mathbf{x}_\nu)$  where one has

$$X_1 < X_2 < \dots < X_\nu < X_{\nu+1} < \dots, \quad L_1 > L_2 > \dots > L_\nu > L_{\nu+1} > \dots$$

Here we should note that

$$L_j \leq X_{j+1}^{-\alpha} \quad (6)$$

for all  $j$  large enough. Of course each vector  $\mathbf{x}_j$  is primitive and each couple  $\mathbf{x}_j, \mathbf{x}_{j+1}$  form a basis of the two-dimensional lattice  $\mathbb{Z}^3 \cap \text{span}(\mathbf{x}_j, \mathbf{x}_{j+1})$ .

Let  $F(\mathbf{x})$  be a linear form linearly independent with  $L$  and  $P$ . Then

$$\max\{|L(\mathbf{x})|, |P(\mathbf{x})|, |F(\mathbf{x})|\} \asymp |\mathbf{x}|. \quad (7)$$

We also use the notation  $P_\nu = P(\mathbf{x}_\nu), F_\nu = F(\mathbf{x}_\nu)$ . In the sequel we need to consider determinants

$$\Delta_j = \begin{vmatrix} L_{j-1} & P_{j-1} & F_{j-1} \\ L_j & P_j & F_j \\ L_{j+1} & P_{j+1} & F_{j+1} \end{vmatrix} = A \begin{vmatrix} x_{0,j-1} & x_{1,j-1} & x_{2,j-1} \\ x_{0,j} & x_{1,j} & x_{2,j} \\ x_{0,j+1} & x_{1,j+1} & x_{2,j+1} \end{vmatrix},$$

here  $A$  is a non-zero constant depending on the coefficients of linear forms  $L, P, F$ . We take into account (7,6) and the inequality  $\alpha > 2$  to see that

$$\Delta_j = L_{j-1}P_jF_{j+1} - L_{j-1}P_{j+1}F_j + O(L_jX_{j+1}^2) = L_{j-1}P_jF_{j+1} - L_{j-1}P_{j+1}F_j + o(1), \quad j \rightarrow \infty. \quad (8)$$

The following statement is a variant of Davenport-Schmidt's lemma. We give it without a proof. It deals with three consecutive minimal points  $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$  lying in a two-dimensional linear subspace, say  $\pi$ . We should note that our definition of minimal points differs from those in [2, 3, 11]. However the main argument is the same. It is discussed in our survey [8]. One may look for the approximation of the one dimensional subspace  $\ell = \pi \cap \{\mathbf{z} : L(\mathbf{z}) = 0\}$  by the points of two-dimensional lattice  $\Lambda_j = \langle x_{j-1}, x_j \rangle$ . Then the points  $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1} \in \Lambda_j$  are the consecutive best approximations to  $\ell$  with respect to the *induced* norm on  $\pi$  (see [8]. Section 5.5).

**Lemma 1.** *If for some  $j$  the points  $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$  are linearly dependent then*

$$\mathbf{x}_{j+1} = t\mathbf{x}_j + \mathbf{x}_{j-1}$$

for some integer  $t$ .

The next statement is known for long time. It comes from Jarník's papers [6, 7]. It was rediscovered by Davenport and Schmidt in [4] and discussed in our survey [8].

**Lemma 2.** *there exist infinitely many indices  $j$  such that the vectors  $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$  are linearly independent.*

The following lemma is due to Jarník [6, 7] (see also Section 5.3 from our paper [8]).

**Lemma 3.** *Suppose that  $j$  is large enough and the points  $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$  are linearly independent. Then*

$$X_{j+1} \gg X_j^{\alpha-1} \quad (9)$$

and

$$L_j \ll X_j^{-\alpha(\alpha-1)} \quad (10)$$

Now we take large  $\nu$  and  $k \geq \nu + 1$  such that

- vectors  $\mathbf{x}_{\nu-1}, \mathbf{x}_\nu, \mathbf{x}_{\nu+1}$  are linearly independent;
- vectors  $\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}$  are linearly independent;
- vectors  $\mathbf{x}_j, \nu \leq j \leq k$  belong to the two-dimensional lattice  $\Lambda_\nu = \mathbb{Z}^3 \cap \text{span}(\mathbf{x}_\nu, \mathbf{x}_{\nu+1})$ .

From Lemma 1 it follows that for  $j$  from the range  $\nu \leq j \leq k-1$  one has

$$L_{j+1} = t_{j+1}L_j + L_{j-1}, \quad P_{j+1} = t_{j+1}P_j + P_{j-1},$$

with some integers  $t_{j+1}$ , and hence

$$L_\nu P_{\nu+1} - L_{\nu+1} P_\nu = \pm(L_{k-1} P_k + L_k P_{k-1}). \quad (11)$$

**Lemma 4.** *Consider positive  $r$  under the condition*

$$r < \alpha^2 - \alpha + 1 < \hat{\omega}^2 - \hat{\omega} + 1. \quad (12)$$

*Suppose that*

$$|P_\nu| \leq L_\nu X_\nu^r \quad (13)$$

*and  $\nu$  is large. Then*

$$|P_{\nu+1}| \gg X_\nu^{\alpha-1}. \quad (14)$$

Proof. For  $j = \nu$  consider the second term in the r.h.s of (8). From (6,7,12.13) and the inequality (9) of Lemma 3 we have

$$|L_{\nu-1} P_\nu F_{\nu+1}| \ll |L_{\nu-1} L_\nu X_\nu^r| X_{\nu+1} \ll X_\nu^{r-\alpha} X_{\nu+1}^{1-\alpha} \ll X_\nu^{r-\alpha^2+\alpha-1} = o(1).$$

As  $\Delta_\nu \neq 0$  we see that

$$1 \ll |L_{\nu-1} P_{\nu+1} F_\nu| \ll L_{\nu-1} |P_{\nu+1}| X_\nu \ll X_\nu^{1-\alpha} |P_{\nu+1}|$$

(in the last inequalities we use (7) and (6). Everything is proved.  $\square$ .)

#### 4. The main estimate.

The following Lemma presents our main argument.

**Lemma 5.** *Suppose that  $r$  satisfies (12). Suppose that (6) holds for all indices  $j$  and suppose that for a certain  $\beta_0$  one has*

$$L_\nu \gg X_\nu^{-\beta_0}. \quad (15)$$

*Suppose that simultaneously we have*

$$|P_\nu| \leq L_\nu X_\nu^r, \quad (16)$$

$$|P_{k-1}| \leq L_{k-1} X_{k-1}^r, \quad (17)$$

$$|P_k| \leq L_k X_\nu^r. \quad (18)$$

*Then*

$$r \geq \alpha^2 + 1 - \frac{\beta_0}{\alpha - 1}. \quad (19)$$

*and*

$$L_k \gg X_k^{-\beta'}, \quad \text{with } \beta' = r - \alpha - 1 + \frac{\beta_0}{\alpha - 1} < \beta_0. \quad (20)$$

First of all we note that

$$L_{\nu+1} |P_\nu| \leq L_\nu L_{\nu+1} X_\nu^r \ll L_\nu X_{\nu+2}^{-\alpha} X_\nu^r \ll L_\nu X_{\nu+1}^{-\alpha} X_\nu^r \ll L_\nu X_\nu^{r-\alpha(\alpha-1)} = o(L_\nu X_\nu^{\alpha-1}).$$

Here the first inequality comes from (16). The second inequality is (6 with  $j = \nu + 1$ . The third one is simply  $X_{\nu+2} \geq X_{\nu+1}$ . The fourth one is (9) of Lemma 3 for  $j = \nu$ . The last inequality here follows from (12) as  $r < \alpha^2 - \alpha + 1 < \alpha^2 - 1$  (because  $\alpha > 2$ ). We see that the conditions of Lemma 4 are satisfied and by Lemma 4 we see that

$$L_\nu |P_{\nu+1}| \gg L_\nu X_\nu^{\alpha-1}.$$

So in the l.h.s. of (11) the first summand is larger than the second. Now from (11) we have

$$L_\nu X_\nu^{\alpha-1} \ll L_{k-1}|P_k| + L_k|P_{k-1}|. \quad (21)$$

We apply (17,18) to see that

$$\max(L_{k-1}|P_k|, L_k|P_{k-1}|) \leq L_{k-1}L_kX_k^r \ll X_k^{r-\alpha}X_{k+1}^{-\alpha} \leq X_k^{r-\alpha^2} \leq X_{\nu+1}^{r-\alpha^2} \ll X_\nu^{(r-\alpha^2)(\alpha-1)}. \quad (22)$$

Here the second inequality comes from (11) for  $j = k - 1$  and  $j = k$ . The third inequality is Lemma 3 with  $j = k$ . The fourth one is just  $X_k \geq X_{\nu+1}$ . The fifth one is Lemma 3 for  $j = \nu$ .

Now from estimates (21,22) and (15) we have

$$X_\nu^{-\beta_0+\alpha-1} \ll X_\nu^{(r-\alpha^2)(\alpha-1)}.$$

This gives

$$r \geq \alpha^2 + 1 - \frac{\beta_0}{\alpha - 1}.$$

So (19) is proved.

To get (20) we combine the estimate (21) with the left inequality of (22), the bound (15) for  $j = \nu$  and the bound (6) for  $j = k - 1$ . This gives

$$X_\nu^{\alpha-1-\beta_0} \leq L_\nu X_\nu^{\alpha-1} \ll L_{k-1}L_kX_k^r \ll L_kX_k^{r-\alpha},$$

or

$$L_k \gg X_k^{\alpha-r}X_\nu^{\alpha-1-\beta_0}.$$

But  $\beta_0 > \alpha(\alpha - 1) \geq (\alpha - 1)$  by inequality (10) of Lemma 3 and  $X_k \geq X_{\nu+1} \gg X_\nu^{\alpha-1}$  by inequality (9) of Lemma 3. So

$$L_k \gg X_k^{\alpha-r+\frac{\alpha-1-\beta_0}{\alpha-1}},$$

and this is the first inequality form (20).

Moreover as  $\beta_0 > \alpha(\alpha - 1)$ , from (12) we deduce  $\beta' < \beta$ . Lemma is proved.  $\square$

### 5. Proof of Theorem 1.

Suppose that  $r$  satisfies (12). We take infinite sequence indices  $\nu_1 < \nu_2 < \dots < \nu_t < \dots$  such that

- for every  $i = 1, 2, \dots$  vectors  $\mathbf{x}_{\nu_{i-1}}, \mathbf{x}_{\nu_i}, \mathbf{x}_{\nu_{i+1}}$  are linearly independent;
- for  $i = 1, 2, \dots$  vectors  $\mathbf{x}_j$ ,  $\nu_i \leq j \leq \nu_{i+1}$  belong to the two-dimensional lattice  $\Lambda_{\nu_i} = \mathbb{Z}^3 \cap \text{span}(\mathbf{x}_{\nu_i}, \mathbf{x}_{\nu_{i+1}})$ .

Now we suppose that three inequalities (16,17,18) hold for all triples  $(\nu, k - 1, k) = (\nu_i, \nu_{i+1} - 1, \nu_{i+1})$  for all  $i \geq 1$ .

Define recursively

$$\beta_{i+1} = r - \alpha - 1 + \frac{\beta_i}{\alpha - 1}.$$

Then

$$\beta_i = \alpha(\alpha - 1) + \frac{\beta_0}{(\alpha - 1)^i} \rightarrow \alpha(\alpha - 1), \quad i \rightarrow \infty.$$

We apply of Lemma 5 to the first  $w$  triple of indices. Then we get (20) for  $k = \nu_{i+1}$ , and in particular for  $k = \nu_w$  with  $\beta_w$  close to  $\alpha(\alpha - 1)$ . Now we apply Lemma 5 to  $\nu = \nu_w$ . in (15) we have  $\beta_w$  instead of  $\beta$ . So (19) gives

$$r \geq \alpha^2 - \alpha + 1 - \frac{\beta_w}{\alpha - 1}$$

We take limit  $w \rightarrow \infty$  to see that

$$r \geq \alpha^2 - \alpha + 1.$$

This contradicts to (12). So there exists  $j \in \cup_{i=1}^{\infty} \{\nu_i, \nu_{i+1} - 1, \nu_{i+1}\}$  such that  $L_j \leq |P_j|X_j^{-r}$ . Theorem is proved.  $\square$

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